$$\frac{x^2+x+1}{(x+1)(x+4)^2} = \frac{A}{x+1} + \frac{B}{x+4} + \frac{C}{(x+4)^2}$$

$$x^2+x+1 = A(x+4)^2 + B(x+1)(x+4) + C(x+1)$$

$\underline{x=-1}$: $\quad 1-1+1 = A(3)^2 \Rightarrow A = 1/9$

$\underline{x=-4}$: $\quad 16-4+1 = C(-3) \Rightarrow C = \frac{-13}{3}$

$\underline{x=0}$: $\quad 1 = A(4)^2 + B(1)(4) + C(1)$

$$1 = \frac{16}{9} + 4B - \frac{13}{3}$$

$$4B = 1 + \frac{13}{3} - \frac{16}{9} = \frac{9+39-16}{9}$$

$$= \frac{32}{9} \Longrightarrow B = \frac{8}{9}.$$

$$\frac{x^2+x+1}{(x+1)(x+4)^2} = \frac{1}{9(x+1)} + \frac{8}{9(x+4)} - \frac{13}{3(x+4)^2}$$

~ partial fraction decomposition done

$$\frac{2x^3+x^2-x-1}{2x^2-x} \leftarrow$$ not proper! apply long division.

$$= x+1 + \underbrace{\frac{-1}{2x^2-x}}$$

$\uparrow$ apply partial fraction to remainder part.

$$\begin{array}{r}
x+1 \\
2x^2-x \,\overline{\big)\, 2x^3+x^2-x-1} \\
-)\ 2x^3-x^2 \qquad\quad \\
\hline
2x^2-x-1 \\
-)\ 2x^2-x \quad\ \\
\hline
-1 \\
\hline
\end{array}$$

$$\frac{-1}{2x^2-x} = \frac{-1}{x(2x-1)}$$

$$= \frac{A}{x} + \frac{B}{2x-1}$$

$$-1 = A(2x-1) + Bx.$$

$\underline{x=0}$: $\quad -1 = A(-1) \Rightarrow A = 1$

$\underline{x=1/2}$: $\quad -1 = B(1/2) \Rightarrow B = -2$

So $$\frac{-1}{2x^2-x} = \frac{1}{x} - \frac{2}{2x-1}$$

$$\frac{2x^3+x^2-x-1}{2x^2-x} = x+1 + \frac{1}{x} - \frac{2}{2x-1}$$

~ partial ~~f~~ fraction decomposition done.

$$\int \frac{x^2+x+1}{(x+1)(x+4)^2}\,dx = \int\left(\frac{1}{9(x+1)} + \frac{8}{9(x+4)} - \frac{13}{3(x+4)^2}\right)dx$$

$$= \frac{1}{9}\ln|x+1| + \frac{8}{9}\ln|x+4| - \frac{13}{3}\int (x+4)^{-2}\,dx$$

$$= \frac{1}{9}\ln|x+1| + \frac{8}{9}\ln|x+4| - \frac{13}{3}\cdot\frac{(x+4)^{-1}}{(-1)(1)} + C \qquad \boxed{\begin{aligned} u &= x+4 \\ du &= dx \end{aligned}}$$

OR

use substitution

$$= \frac{1}{9}\ln|x+1| + \frac{8}{9}\ln|x+4| + \frac{13}{3}(x+4)^{-1} + C$$

$$\int \frac{2x^3+x^2-x-1}{2x^2-x}\,dx = \int\left(x+1+\frac{1}{x} - \frac{2}{2x-1}\right)dx$$

$$= \frac{x^2}{2} + x + \ln|x| - \frac{\cancel{2}}{\cancel{2}}\ln|2x-1| + C$$

$$= \frac{1}{2}x^2 + x + \ln|x| - \ln|2x-1| + C$$

$$\int \frac{x-1}{(x^2+1)x}\,dx$$

$$\frac{x-1}{(x^2+1)x} = \frac{Ax+B}{x^2+1} + \frac{C}{x}$$ ← apply partial fraction decomposition first.

$$x-1 = (Ax+B)x + C(x^2+1)$$

$\underline{x=0:}$ $-1 = 0 + C(1) \Rightarrow C = -1$

$$\underbrace{x-1}_{\downarrow} = (Ax+B)x - (x^2+1)$$

$$= Ax^2 + Bx - x^2 - 1$$

$$0 \cdot x^2 + x - 1 = (A-1)x^2 + Bx - 1$$

Comparing coefficients of the polymails on the Left and Right handside of the equation above:

Coefficient of x^2 : $0 = A-1 \Rightarrow A = 1$

" " x : $1 = B$

$$\frac{x-1}{(x^2+1)x} = \frac{x+1}{x^2+1} - \frac{1}{x}$$

$$\int \frac{x-1}{(x^2+1)x}\,dx = \int \underbrace{\frac{x}{x^2+1}} + \frac{1}{x^2+1} - \frac{1}{x}\,dx.$$

$u = x^2+1$

$du = 2x\,dx$

$$= \int \frac{1}{u}\cdot\frac{1}{2}\,du + \int \frac{1}{x^2+1}\,dx - \int \frac{1}{x}\,dx$$

$$= \frac{1}{2}\ln|u| + \arctan(x) - \ln|x| + C$$

$$= \frac{1}{2}\ln(x^2+1) + \arctan(x) - \ln|x| + C.$$